Admittance of rugate filters derived from a 2 × 2 inhomogeneous matrix

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Admittance diagrams are used to analyze qualitatively and quantitatively the behavior of inhomogeneous thin films with an arbitrary refractive-index function. From this study it follows that one can treat the behavior of rugate filters by using the concept of effective refractive indices, which are associated with the phase integral in a simple way. These results are applied to the study of periodic systems, and, as expected, one can consider high and low effective refractive indices to determine the important parameters of these stop bands. With these ideas it is possible that one can deal with rugate filters more closely as homogeneous periodic systems by taking advantage of the existing theory.

Key words: Rugate filters, inhomogeneous thin films, admittance.

Introduction

It is well known that inhomogeneous thin films with a sinusoidal refractive-index function behave much like periodic homogeneous multilayers, and in fact many scientists have exploited these similarities to develop a wide variety of stop bands. Consequently rugates have been studied extensively in the past[1-6] and currently they constitute one of the most important applications[3,6] of inhomogeneous thin films. On this subject Southwell[7] applied the coupled-mode theory to develop a method to analyze the spectral performance of this kind of filter. His approach gave good results when the refractive-index function had a small amplitude compared with the average refractive index. With this theory he derived simple expressions to calculate the characteristic parameters of the stop band.

Recently Bovard[8] developed a generalization of the classical 2 × 2 matrix theory for homogeneous systems to study important properties and parameters of inhomogeneous systems in general. From this theory the Fourier-transform synthesis technique comes naturally, and although it is approximated it constitutes a powerful tool for thin-film design.

In this study we establish arguments to analyze the close relationship between homogeneous multilayers and inhomogeneous thin films. Using the matrix theory developed by Bovard and the admittance construction,[9] we find the intersections of the admittance curve with the real axis to calculate the effective refractive indices.

This concept is applied to a rugate filter, and we demonstrate that very simple expressions can be derived to determine the width, height, and wavelength shift of the stop band under nonnormal incidence. Finally we show that these results agree with those obtained by Southwell[7] when the modulation of the refractive-index function is small.

Theory

Admittance of an Inhomogeneous System

Following the matrix formalism for propagation of the electromagnetic field through an inhomogeneous nonabsorbing thin film[8] (Fig. 1), we see that the admittance Y of a thin film of thickness z is involved in the characteristic matrix as

$$E_i\begin{pmatrix}1\\Y\end{pmatrix}=\begin{pmatrix}\left[\dfrac{\eta(\beta)}{\eta(0)}\right]^{1/2}L(\beta) & \dfrac{iG(\beta)}{[\eta(\beta)\eta(0)]^{1/2}} \\ i[\eta(\beta)\eta(0)]^{1/2}K(\beta) & \left[\dfrac{\eta(0)}{\eta(\beta)}\right]^{1/2}F(\beta)\end{pmatrix}\begin{pmatrix}1\\Y'\end{pmatrix}E_s. \quad (1)$$

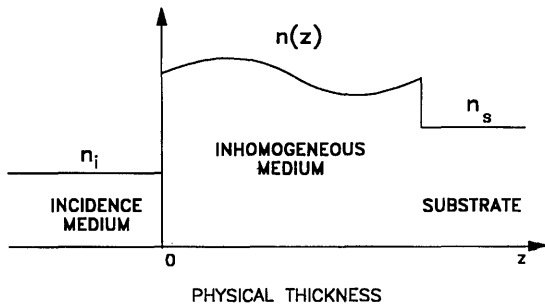Here E_i and E_s are the electric fields in the inci-

Fig. 1. Refractive index of the inhomogeneous thin film as a function of physical thickness.

dence medium and the substrate, respectively, and β is the phase integral:

$$\beta(z) = \frac{2\pi}{\lambda} \int_0^z n(z)\cos\theta(z)\mathrm{d}z, \qquad (2)$$

where $n(z)$ is the index of refraction of the film, $\theta(z)$ is the propagation angle, and λ is the wavelength. The total admittance of the substrate plus the thin film at a given thickness is represented by the complex quantity $Y' = \alpha + i\gamma$. The angular dependence of the admittance of the film at a given point z is

$$\eta(z) = \begin{cases} \eta_v n(z)\cos\theta(z) & \text{case TE} \\ \eta_v n(z)/\cos\theta(z) & \text{case TM} \end{cases}, \qquad (3)$$

where η_v is the admittance of the vacuum space.

From Eq. (1) we can obtain the admittance equation, which gives the evolution of the admittance as the film grows:

$$Y = x + iy = \frac{\left(\frac{\eta_0}{\eta}\right)^{1/2}F\alpha + i\left[(\eta_0\eta)^{1/2}K + \left(\frac{\eta_0}{\eta}\right)^{1/2}F\gamma\right]}{\left[\left(\frac{\eta}{\eta_0}\right)^{1/2}L - (\eta_0\eta)^{1/2}G\gamma\right] + i(\eta_0\eta)^{1/2}G\gamma}. \qquad (4)$$

Here we call $\eta_0 = \eta(0)$ and omit the explicit dependence on β.

Functions $F(\beta)$, $K(\beta)$, $G(\beta)$, and $L(\beta)$ are expressed in terms of another set of functions by

$$F(\beta) = f(\beta)\cos\beta + k(\beta)\sin\beta,$$
$$K(\beta) = f(\beta)\sin\beta - k(\beta)\cos\beta,$$
$$G(\beta) = l(\beta)\sin\beta + g(\beta)\cos\beta,$$
$$L(\beta) = l(\beta)\cos\beta - g(\beta)\sin\beta. \qquad (5)$$

The functions $f(\beta)$, $k(\beta)$, $g(\beta)$, and $l(\beta)$, which are related to the index profile through an integral equation, are in Appendix A.

Using Eqs. (5) in Eq. (4) and separating the real and imaginary parts and using them to eliminate the phase parameter β, after some algebra we can obtain the equation that determines the admittance locus:

$$\frac{\alpha}{\eta_0}(g^2 + l^2)(x^2 + y^2) - \frac{\alpha^2 + \gamma^2 + \eta^2}{\eta}x$$
$$+ 2\alpha(gf + lk)y + (f^2 + k^2)\alpha\eta_0 = 0. \qquad (6)$$

If all parameters were constant except x and y, Eq. (6) would represent circles with their centers on

$$\left[\frac{\eta_0(\alpha^2 + \gamma^2 + \eta^2)}{2\alpha\eta(g^2 + l^2)}, -\frac{\eta_0(gf + lk)}{g^2 + l^2}\right] \qquad (7)$$

and radii at

$$r_a = \frac{\eta_0[(\alpha^2 + \gamma^2 + \eta^2)^2 - 4\alpha^2\eta^2]^{1/2}}{2\alpha\eta(g^2 + l^2)}. \qquad (8)$$

However, functions g, f, l, and k change with phase β, and the radius changes too as the film grows. Even though we cannot predict the behavior of these functions analytically, it is possible to establish some restrictions to analyze some interesting points: the intersections of the admittance curve with the real axis, for example. To be sure that the curve intersects the real axis, we can demonstrate that the condition

$$|gf + lk| \leq \frac{[(\alpha^2 + \gamma^2 + \eta^2)^2 - 4\alpha^2\eta^2]^{1/2}}{2\alpha\eta} \qquad (9)$$

must be satisfied.

When the coupling condition $\eta(0) = \eta_i$ and $\eta(\beta) = \eta_s$ is satisfied, it can be demonstrated that

$$gf + lk = 0 \qquad (10)$$

for the intersections with the real axis. Assuming that the substrate has no absorption, setting $\gamma = 0$ in Eq. (6), and taking into account that $\alpha(0) = \eta_s$, $l(0) = 1$, and $g(0) = 0$, we give the intersections as

$$x_0 = \eta_s, \qquad (11)$$

$$x_j = \frac{\eta\eta_0}{x_{j-1}(g^2 + l^2)}, \qquad j = 1, 2, 3, \dots. \qquad (12)$$

Here x_0 is the starting point on the curve where $\beta = 0$ and x_j shows all the subsequent intersections with the real axis as the phase thickness increases.

Now let us consider the reflection coefficient

$$\rho = \frac{\eta_i - Y}{\eta_i + Y}. \qquad (13)$$

Considering the coupling condition given above, we can demonstrate[10] that

$$|\rho|\sin\varphi_\rho = \frac{2(gf + lk)}{(f + l)^2 + (k - g)^2}, \qquad (14)$$
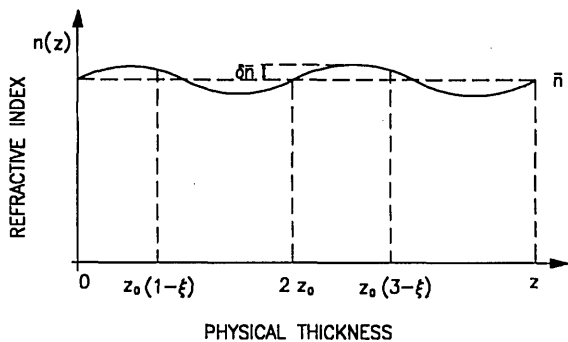
Fig. 2. Periodic refractive-index function with period $\lambda_0/2\bar{n}$. Here $z_0 = \lambda/4\bar{n}$. The quantity ξ is a small number that identifies the points that correspond to optical thicknesses of odd multiples of $\lambda_0/4$.
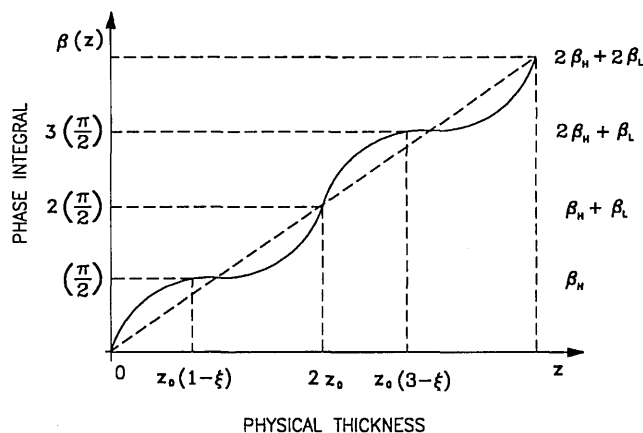

Fig. 3. Phase integral for a period refractive-index function, which is also a periodic but increasing curve that oscillates around the oblique line whose slope depends on the magnitude of the average refractive index.

where φ_ρ represents the change in phase on reflection. If we consider the condition of the intersections given in Eq. (10), we find from Eq. (14) that

$$\varphi_\rho = m\pi, \qquad m = 0, 1, 2 \ldots, \qquad (15)$$

which means that whenever the optical thickness is a multiple of a quarter of a wavelength

$$\int_0^z n(z)\cos\theta(z)\mathrm{d}z = m\frac{\lambda}{4}. \qquad (16)$$

The admittance curve intersects the real axis.

Admittance of Periodic Systems

The matrix formalism permits us to treat a stack of thin films by taking the product of the characteristic matrices of each component film to obtain the total response of the system. For an inhomogeneous thin film with a periodic refractive-index function around a mean value $\bar{n}$ (Fig. 2), it is possible to calculate its

Here δ is the relative fluctuation of the refractive-index function about the average value $\bar{n}$. With this design we can calculate the phase integral in the interval $[z_1, z_2]$ from Eq. (2) to obtain (Fig. 3)

$$\beta = \left(\frac{2\pi}{\lambda_0}\bar{n}z + \frac{\delta}{2}\cos\frac{4\pi}{\lambda_0}\bar{n}z\right)_{z_1}^{z_2}. \qquad (18)$$

Whenever the optical thickness is a multiple of $\lambda_0/2$, the refractive index takes the initial mean value $\bar{n}$, and we can define a period that will be replicated a number of times. Every period is divided into two slices with equal optical thicknesses $\lambda_0/4$. We may then treat this configuration by using the associated characteristic matrix to study the total behavior of such a periodic inhomogeneous system with p periods:

$$E_i\begin{pmatrix}1\\Y\end{pmatrix} = \left\{\begin{pmatrix}\left[\frac{n(\beta_H)}{n(0)}\right]^{1/2}L(\beta_H) & \frac{iG(\beta_H)}{[n(\beta_H)n(0)]^{1/2}}\\ i[n(\beta_H)n(0)]^{1/2}K(\beta_H) & \left[\frac{n(0)}{n(\beta_H)}\right]^{1/2}F(\beta_H)\end{pmatrix}\begin{pmatrix}\left[\frac{n(\beta_H+\beta_L)}{n(\beta_H)}\right]^{1/2}L(\beta_L) & \frac{iG(\beta_L)}{[n(\beta_H+\beta_L)n(\beta_H)]^{1/2}}\\ i[n(\beta_H+\beta_L)n(\beta_H)]^{1/2}K(\beta_L) & \left[\frac{n(\beta_H)}{n(\beta_H+\beta_L)}\right]^{1/2}F(\beta_L)\end{pmatrix}\right\}^p\begin{pmatrix}1\\n_s\end{pmatrix}E_s, \quad (19)$$

spectral performance easily by taking advantage of the periodicity of the system. We proceed to divide the film into slices, identifying periods as in the case of homogeneous multilayer stacks. The product of the characteristic matrices that correspond to each period yields the optical performance.

For simplicity let us consider a film under normal incidence with the refractive-index function defined in Ref. 7 in terms of the physical thickness z:

$$n(z) = \bar{n}\left(1 + \delta\sin\frac{4\pi}{\lambda_0}\bar{n}z\right). \qquad (17)$$

where the dependence on β_H and β_L of the functions $L, G, H,$ and K indicates that they must be evaluated within the limits from zero to β_H or from β_H to β_L, respectively.

The admittance loci for any periodic profile are open curves because of the index profile variation as shown in Fig. 4; a useful parameter that can be obtained from this diagram is the intersection of the admittance curve with the real axis, which corresponds to the optical thickness equal to $m\lambda_0/4$, where m is an integer. Rewriting Eqs. (11) and (12), we
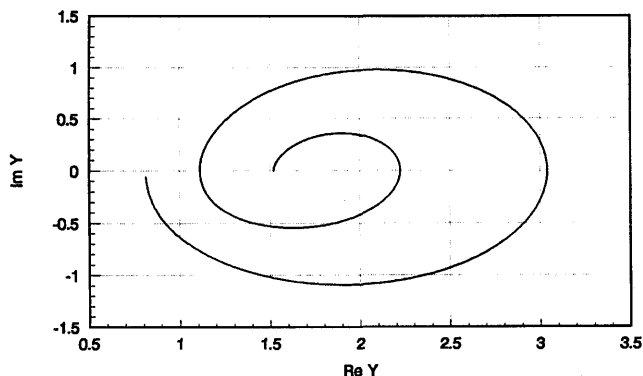
Fig. 4. Admittance of the rugate filter at the wavelength of the design. The curve intersections correspond to optical thickness multiples of $\lambda_0/4$.
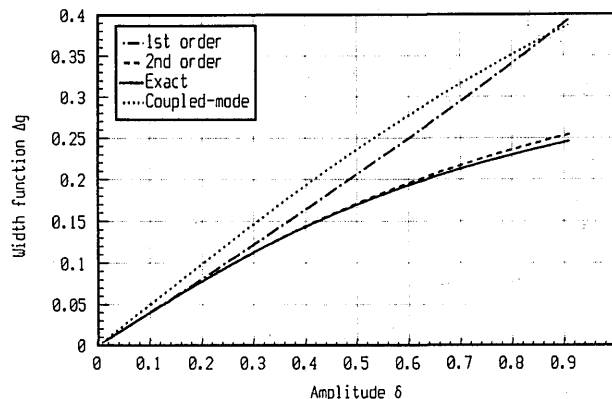

Fig. 5. Width function Δg of a stop band with a sinusoidal refractive-index function versus amplitude δ. The parameters in Eq. (17) are $\bar{n} = 3.5$ and $\lambda_0 = 1000$ nm.

find that

$$x_0 = n_s, \tag{20}$$

$$x_j = \frac{n(0)n(\beta)}{x_{j-1}[g^2(\beta) + f^2(\beta)]}, \qquad j = 1, 2, 3, \ldots \tag{21}$$

Considering the dependence on β_H or β_L, we can call

$$n_H{}^2 = \frac{n(0)n(\beta_H)}{g^2(\beta_H) + f^2(\beta_H)},$$

$$n_L{}^2 = \frac{n(\beta_H)n(\beta_H + \beta_L)}{g^2(\beta_L) + f^2(\beta_L)}, \tag{22}$$

so that intersections for normal incidence can be expressed as

$$x_j = \begin{cases} n_s\left(\dfrac{n_L}{n_H}\right)^j & j = 0, 2, 4, \ldots \\[3mm] \dfrac{n_H{}^{j+1}}{n_s n_L{}^{j-1}} & j = 1, 3, 5, \ldots \end{cases} \tag{23}$$

These intersections represent the turning points in the reflectivity of the system, interpreted now as a stack of homogeneous layers.

The calculation of the effective indices in this stack is directly related to the phase integral. From Eq. (16) we obtain

$$n_H = \frac{1}{\Delta_H} \int_0^{\Delta_H} n(z)\mathrm{d}z, \tag{24}$$

$$n_L = \frac{1}{\Delta_L} \int_0^{\Delta_L} n(z - \Delta_H)\mathrm{d}z, \tag{25}$$

where Δ_H and Δ_L represent the physical thicknesses associated with two slices with optical thicknesses $\lambda_0/4$ of high and low effective index, respectively, and we can show that

$$\Delta_H = \frac{\lambda_0}{4\bar{n}}(1 - \xi), \tag{26}$$

$$\Delta_L = \frac{\lambda_0}{4\bar{n}}(1 + \xi), \tag{27}$$

where ξ is defined as

$$\xi = \frac{\delta}{\pi}[1 + \cos(\pi\xi)], \tag{28}$$

which represents the exact solution. Taking the first and second terms of the cosine series expansion,

$$\xi = \frac{2\delta}{\pi}, \tag{29}$$

$$\xi = \frac{-1 + (1 + 4\delta^2)^{1/2}}{\pi\xi}, \tag{30}$$

we obtain the first- and second-order approximations.

We can now calculate the number of periods that we need to reach a given reflectance R_0 with this periodic system:

$$p = \frac{\ln\left\{\dfrac{n_i}{n_s}\left[\dfrac{1 - (R_0)^{1/2}}{1 + (R_0)^{1/2}}\right]\right\}}{2\ln\left(\dfrac{n_L}{n_H}\right)} \tag{31}$$
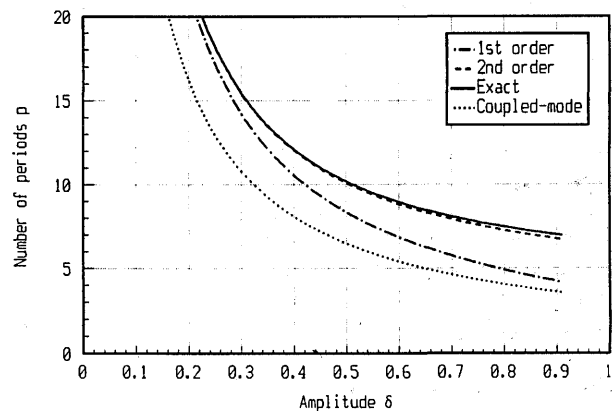

Fig. 6. Number of periods that are needed to obtain a reflectance of $R_0 = 99.99\%$ for the wavelength of the design in Fig. 5.
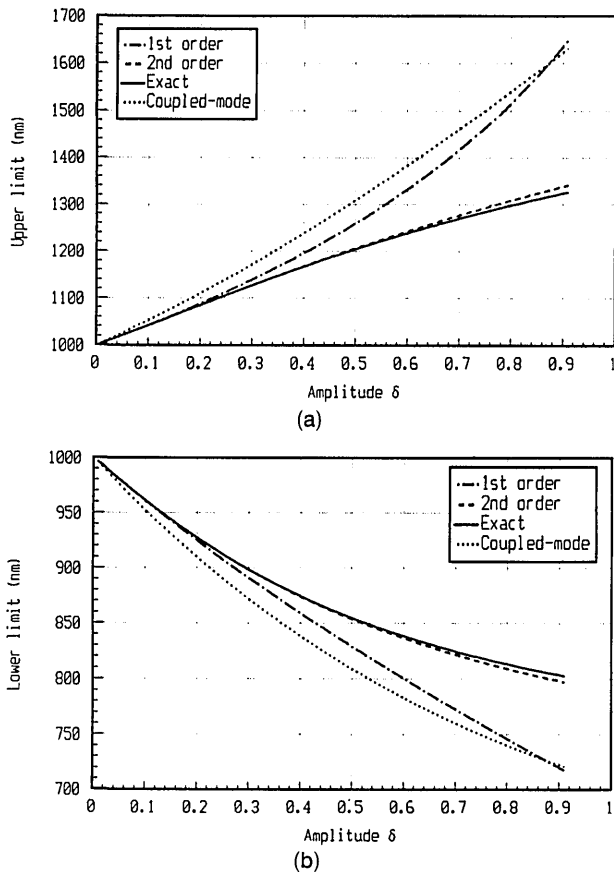
(a)

(b)

Fig. 7. Wavelengths of the stop band: (a) upper limit, (b) lower limit.

and the width of the stop band[11]

$$\Delta g = \frac{2}{\pi} \arcsin\left(\frac{n_H - n_L}{n_H + n_L}\right). \qquad (32)$$

In this case $g = \lambda_0/\lambda$ is different from g in Eqs. (5). Figures 5 and 6 show the behavior of Δg and p for the different approximations, and we include for comparison the results Southwell obtained with the coupled-wave theory. Figure 7 shows the edges of the stop band in terms of the wavelength.

The shift in wavelength of the stop band under nonnormal incidence can be obtained from Eq. (2) as

$$\int_0^z n(z)\cos \theta(z)\mathrm{d}z \approx \int_0^z n\left(1 - \frac{n_i^2 \sin^2 \theta_i}{n^2}\right)\mathrm{d}z. \qquad (33)$$

The wavelength shift in Southwell's result can be found with the equation for small amplitude fluctuations in the refractive index, that is, $\delta \ll 1$ in Eq. (17); then

$$\Delta\lambda \approx \left(\frac{n_i^2 \sin^2 \theta_i}{2\bar{n}^2}\right)\lambda_0. \qquad (34)$$

Conclusion

It has been demonstrated that admittance diagrams can be helpful in the analysis of inhomogeneous thin films since they show a straightforward picture of their behavior. It is also shown that it is possible to treat an inhomogeneous film as a homogeneous multilayer stack with periods formed by two effective refractive indices, high and low. We have calculated for a proposed system the number of periods required for a given reflectivity, the width, and the shift of the stop band under nonnormal incidence. These are the basic parameters for this kind of systems. Our results are compared with those reported in the literature previously.

Extensions of this method to other multilayer systems such as narrow-band filters and multiline stop bands are the subjects of future research.

Appendix A: Integral Functions

Functions $f(\beta)$, $g(\beta)$, $k(\beta)$, and $l(\beta)$ are a series given by

$$f(\beta) = 1 + C_1(\beta) + C_2(\beta) + \cdots,$$
$$g(\beta) = S_1(\beta) + S_2(\beta) + S_3(\beta) + \cdots,$$
$$l(\beta) = 1 - C_1(\beta) + C_2(\beta) - \cdots,$$
$$k(\beta) = S_1(\beta) - S_2(\beta) + S_3(\beta) - \cdots, \qquad (A1)$$

where $C_m(\beta)$ and $S_m(\beta)$ are the family of integral functions:

$$C_1(\beta) = \int_0^\beta r(\beta_1)\cos 2\beta_1 \mathrm{d}\beta_1, \qquad (A2)$$

$$S_1(\beta) = \int_0^\beta r(\beta_1)\sin 2\beta_1 \mathrm{d}\beta_1, \qquad (A3)$$

$$C_2(\beta) = \int_0^\beta \int_0^{\beta_1} r(\beta_1)r(\beta_2)\cos 2(\beta_2 - \beta_1)\mathrm{d}\beta_2\mathrm{d}\beta_1, \quad (A4)$$

$$S_2(\beta) = \int_0^\beta \int_0^{\beta_1} r(\beta_1)r(\beta_2)\sin 2(\beta_2 - \beta_1)\mathrm{d}\beta_2\mathrm{d}\beta_1, \quad (A5)$$

$$C_m(\beta) = \int_0^\beta \int_0^{\beta_1} \cdots \int_0^{\beta_{m-1}} r(\beta_1)r(\beta_2)\ldots r(\beta_m)$$
$$\times \cos 2(\beta_m - \beta_{m-1} + \cdots + \cdots \beta_1)$$
$$\times \mathrm{d}\beta_m \ldots \mathrm{d}\beta_2\mathrm{d}\beta_1, \qquad (A6)$$

$$S_m(\beta) = \int_0^\beta \int_0^{\beta_1} \cdots \int_0^{\beta_{m-1}} r(\beta_1)r(\beta_2)\ldots r(\beta_m)$$
$$\times \sin 2(\beta_m - \beta_{m-1} + \cdots + \cdots \beta_1)$$
$$\times \mathrm{d}\beta_m \ldots \mathrm{d}\beta_2\mathrm{d}\beta_1, \qquad (A7)$$

$$r(\beta) = \frac{\eta'(\beta)}{2\eta(\beta)}. \qquad (A8)$$

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